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MANIPULATION OF QUANTUM EVOLUTION

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Abstract

The free evolution of a non–relativistic charged particle is manipulated using time–dependent magnetic fields. It is shown that the application of a programmed sequence of magnetic pulses can invert the free evolution process, forcing an arbitrary wave packet to "go back in time" to recover its past shape. The possibility of more general operations upon the Schrödinger wave packet is discussed.

1 Introduction

We expect that in a near future the problem of particle trapping will be replaced by a wider manipulation problem concerning the purposeful operations on quantum states. This involves the inverse evolution problem: given a unitary operator acting in the Hilbert space of states of a quantum system, one asks if there exists a realistic (possibly time–dependent) Hamiltonian inducing this operator as a result of a dynamical evolution process. The importance of the subject: the unitary operations which can be dynamically induced, can also be used to control the wave–like behaviour of quantum objects, e.g. during the preparation of a measurement. The so defined subject was put forward by Lamb [1] and it has been subsequently developed by his followers (Lubkin [2], Mielnik [3, 4], Royer [5], Brown [6], Fernández [7], and other authors.) A key to the manipulation of a quantum state lies in the possibility of trapping the particle in a circular dynamical process called an *evolution loop* [4] (EL). In an EL the evolution operator $U(t)$ becomes the identity for a finite time interval. The subsequent perturbation of the EL can induce arbitrary unitary operations on the wave packet as the result of the cummulative process involving the small precessions of the distorted loop [3].

The unitary transformations that will be discussed in this paper are:

$$\begin{cases} \text{evolution loop} & U(\tau) = \mathbf{1}, \ \tau > 0, \\ \text{rigid displacement of the wave packet} & U(\tau) = e^{i\mathbf{a}\cdot\mathbf{p}/\hbar}, \\ \text{the quantum time machine} & U(\tau) = e^{-iT'\mathbf{p}^2/2m\hbar}, \ -\infty < T' < \infty. \end{cases}$$

2 The evolution loops

Consider a non-relativistic spinless particle of charge e evolving under the action of a homogeneous time-dependent magnetic field $\mathbf{B}(t)$. The Hamiltonian of the system can be expressed as:

$$H(t) = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{2c}\mathbf{r}\times\mathbf{B}(t)\right)^2 = \frac{1}{2m}\left[\mathbf{p}^2 + \left(\frac{e\mathbf{B}(t)}{2c}\right)^2 \mathbf{r}_\perp^2\right] - \frac{e\mathbf{B}(t)\cdot\mathbf{L}}{2mc}, \tag{1}$$

where $\mathbf{L}$ is the angular momentum operator and $\mathbf{r}_\perp = \mathbf{r} - \left(\mathbf{r}\cdot\frac{\mathbf{B}(t)}{|\mathbf{B}(t)|}\right)\frac{\mathbf{B}(t)}{|\mathbf{B}(t)|}$ is the part of $\mathbf{r}$ orthogonal to $\mathbf{B}(t)$. Here, $\mathbf{B}(t)$ will be taken as the sequence of identically shaped orthogonal pulses in the three directions $x_1,\ x_2,\ x_3$ defined by the right-handed orthonormal set of basis vectors $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$:

$$\mathbf{B}(t) = \begin{cases} B(t)\mathbf{e_1} & \text{for } t\in[0,T), \\ B(t-T)\mathbf{e_2} & \text{for } t\in[T,2T), \\ B(t-2T)\mathbf{e_3} & \text{for } t\in[2T,3T), \end{cases} \tag{2}$$

with $B(t) = B\beta(t/T)$, $\int_0^1 \beta(t')dt' = 0$. The generic evolution can be determined through the operator $U(3T,0) = U(3T,2T)U(2T,T)U(T,0)$. In dimenssionless units ($t' = t/T$, $\tilde{\mathbf{q}} = \sqrt{m/\hbar T}\ \mathbf{q}$, $\tilde{\mathbf{p}} = \sqrt{T/m\hbar}\ \mathbf{p}$) it takes the form:

$$\begin{aligned} U(t'=3,0) &= W_1W_2W_3, \\ W_1 &= \Omega(\tilde{x}_1,\tilde{p}_1)^2\exp(-i\tilde{p}_1^2/2), \\ W_2 &= \Omega(\tilde{x}_2,\tilde{p}_2)\exp(-i\tilde{p}_2^2/2)\Omega(\tilde{x}_2,\tilde{p}_2), \\ W_3 &= \exp(-i\tilde{p}_3^2/2)\Omega(\tilde{x}_3,\tilde{p}_3)^2. \end{aligned} \tag{3}$$

Above, $\Omega(q,p) = \mathcal{T}\left\{\exp\left(-i\int_0^1 \tilde{H}(t')dt'\right)\right\}$ is the evolution operator induced by the one-dimensional harmonic oscillator of variable frequency $\alpha(t') = \left(\frac{eBT}{2mc}\right)\beta(t') = \alpha\beta(t')$ and Hamiltonian $\tilde{H}(t') = p^2/2 + \alpha(t')^2q^2/2$, q and p are two canonically conjugated operators such that $[q,p] = i$ and $\mathcal{T}$ is the time ordering operator.

Now, because W_i depends on Hamiltonians which are quadratic in the canonical variables $(\tilde{x}_i,\tilde{p}_i)$ but it doesn't involve $(\tilde{x}_j,\tilde{p}_j)$ $j\neq i$, it is possible to represent it by a 2×2 matrix w_i (the "Heisenberg picture"):

$$W_i^\dagger \begin{bmatrix}\tilde{x}_i\\ \tilde{p}_i\end{bmatrix} W_i = w_i\begin{bmatrix}\tilde{x}_i\\ \tilde{p}_i\end{bmatrix}, \tag{4}$$

where $i = 1,2,3$. The kind of dynamical process (3) depends on the algebraic type of the matrices w_i, and due to the form of the operators W_i it is determined just by one c-number invariant called *the discriminant*:

$$\Delta(\alpha\beta) = Tr(w_1) = Tr(w_2) = Tr(w_3). \tag{5}$$

Whenever this invariant accepts one of the distinguished values:

$$\Delta(\alpha\beta) = 2\cos\frac{2\pi l}{n},\quad l,n = \pm1,\pm2,\cdots, \tag{6}$$

the matrices w_i fulfill the algebraic identity $w_i^n = 1$ [4] $\Rightarrow$ $W_i^n = \mathbf{1}$. Hence:

$$U(t' = 3n, 0) = U(3,0)^n = \mathbf{1}. \tag{7}$$

It can be shown [8] that for any piece–wise continuous bounded real function $\beta(t')$, $0 \le t' \le N$ with N finite, there exist pulse amplitudes α for which the discriminant $\Delta(\alpha\beta)$ accepts any of the special values (6). Whenever this happens, the n repetitions of our magnetic pulse pattern (2) generate the evolution loops in the space of states $L^2(\mathbf{R}^3)$ at the loop period $\tau = 3nT$.

As an illustration we restrict the discussion to the case of rectangular pulses:

$$\beta(t') = \theta(1/2 - t')\theta(t') - \theta(t' - 1/2)\theta(1 - t'), \tag{8}$$

where $\theta(x)$ is the step function. In this case, the discriminant $\Delta(\alpha\beta)$ can be analytically determined taking the form $\Delta(\alpha\beta) = 2\cos 2\alpha - \alpha \sin 2\alpha$. The simplest EL is achieved making $n = 4$, $l = 1$ in (6); the solution for α becomes:

$$\alpha = 0.632295\cdots. \tag{9}$$

The loop period is $\tau = 12T$, and the orders of magnitude of the field strenght and T must satisfy the relation $B = 2\alpha mc/eT$ with the value of α in (9).

3 Rigid displacement of the wave packet

The evolution loops provide a convenient method to generate arbitrary unitary transformations of quantum states. Suppose, e.g., we want to produce a rigid displacement of the wave packet. To this end, we take the loop induced by the rectangular pulses (8) with the α–value (9) as the unperturbed system. The loop is then perturbed by a homogeneous time–dependent external force $\mathbf{F}(t) = e\mathbf{E}(t)$. The total Hamiltonian becomes $H(t) = H_0(t) - e\mathbf{r} \cdot \mathbf{E}(t)$, where $H_0(t)$ is the loop part and $-e\mathbf{r} \cdot \mathbf{E}(t)$ is the perturbation. The evolution operator within one loop period $\tau = 12T$ can be evaluated in the interaction picture. Hence:

$$U(\tau) = \exp\left(\frac{i}{\hbar}\int_0^\tau \mathbf{F}(t) \cdot \mathbf{r}_0(t) dt\right) = \exp\left[\frac{i}{\hbar}(\mathbf{a} \cdot \mathbf{p} + \mathbf{b} \cdot \mathbf{r})\right], \tag{10}$$

where $\mathbf{r}_0(t)$ is the triplet of canonical operators $x_1(t)$, $x_2(t)$, $x_3(t)$ in the Heisenberg frame of $H_0(t)$. By taking $\mathbf{F}(t) = \mathbf{F}\sin 2\pi t/\tau$ it is possible to obtain explicitly $\mathbf{a}$ and $\mathbf{b}$ in (10) [8]. With the aim of produce the pure rigid displacement, with $\mathbf{b} = 0$, we selected $\mathbf{F}(t)$ as a sequence of pulses of rectangular force. It was found that a single pulse of rectangular force $\mathbf{F} = e\mathbf{E}$ in the x_1–direction acting within the interval $[9T, 10T]$ displaces the packet *against* the applied force by $a_x = -1.658693FT^2/m$ (the *boomerang effect*). Other possibilities are discussed elsewhere [8].

4 The quantum time machine

As the next example we shall discuss the *quantum time machine*. In this scheme, the acceleration, slowing or inversion of the free evolution of the charged particle is possible. Once again, the technology is based on a sequence of pulses of homogeneous magnetic field of the form:

$$\mathbf{B}(t) = \begin{cases} B(t)\mathbf{e_1} & \text{for } t \in [0, 2T), \\ B(t-2T)\mathbf{e_2} & \text{for } t \in [2T, 4T), \\ B(t-4T)\mathbf{e_3} & \text{for } t \in [4T, 6T), \quad \int_0^{2T} B(t)dt = 0, \end{cases} \tag{11}$$

with $B(t)$ given by:

$$B(t) = \begin{cases} B_1 & \text{for } t \in [0, t_1), \qquad (0 < t_1 < T) \\ B_2 & \text{for } t \in [t_1, T), \\ -B_2 & \text{for } t \in [T, T+t_2), \ (t_2 = T - t_1), \\ -B_1 & \text{for } t \in [T+t_2, 2T). \end{cases} \tag{12}$$

The key evolution operator $U(\tau = 6T, 0)$ in dimensionless coordinates and time $t' = t/T$ takes a similar form as in (3):

$$U(6,0) = \Omega^2 G \otimes \Omega G \Omega \otimes G \Omega^2, \tag{13}$$

where $\Omega = \Omega(2)$ and $G = G(2)$ describe the evolution of the oscillator of variable frequency and the free evolution in $[0, 2]$ respectively. All the calculations are made in the matrix representation, working with the angular parameters $\gamma_1 = \alpha_1 t_1'$ and $\gamma_2 = \alpha_2 t_2'$, where $\alpha_1 = eB_1T/2mc$, $\alpha_2 = eB_2T/2mc$, $t_1' = t_1/T$, $t_2' = t_2/T$. It turns out that when the amplitudes and times of the pulses of magnetic field satisfy the relations [8]:

$$\begin{aligned} \alpha_1 &= \frac{\gamma_1 \tan\gamma_1 - \gamma_2 \tan\gamma_2}{\tan\gamma_1}, \\ \alpha_2 &= \frac{\gamma_1 \tan\gamma_1 - \gamma_2 \tan\gamma_2}{-\tan\gamma_2}, \\ t_1' &= \frac{\gamma_1 \tan\gamma_1}{\gamma_1 \tan\gamma_1 - \gamma_2 \tan\gamma_2}, \\ t_2' &= \frac{-\gamma_2 \tan\gamma_2}{\gamma_1 \tan\gamma_1 - \gamma_2 \tan\gamma_2}, \end{aligned} \tag{14}$$

the free evolution operator is produced at the end time of the sequence (11):

$$U(\tau) = \exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}T'\right) = \exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}\tau\chi\right), \tag{15}$$

where the "effective" time $T' = \tau\chi = 6T\chi$ depends on the *distorsion coefficient*:

$$\chi = \frac{1}{3} + \frac{2}{3}\cos^2\gamma_2 \frac{\tan^2\gamma_1 - \tan^2\gamma_2}{\gamma_1 \tan\gamma_1 - \gamma_2 \tan\gamma_2}. \tag{16}$$

As from definition t_1' and t_2' must be positive, then γ_1 and γ_2 must lie in intervals of different parity, i.e. $n\pi < \gamma_1 < (n + 1/2)\pi$ and $(m - 1/2)\pi < \gamma_2 < m\pi$, $m, n \in \mathbf{Z}^+$ or vice versa. The distorsion coefficient χ as function of γ_1 and γ_2 is plotted in the Fig.1. As can be seen, one can generate three different situations:

$$\begin{cases} \chi > 1 & \text{free evolution acceleration} \\ 0 < \chi \le 1 & \text{free evolution slowing} \\ \chi \le 0 & \text{free evolution regression} \end{cases}$$

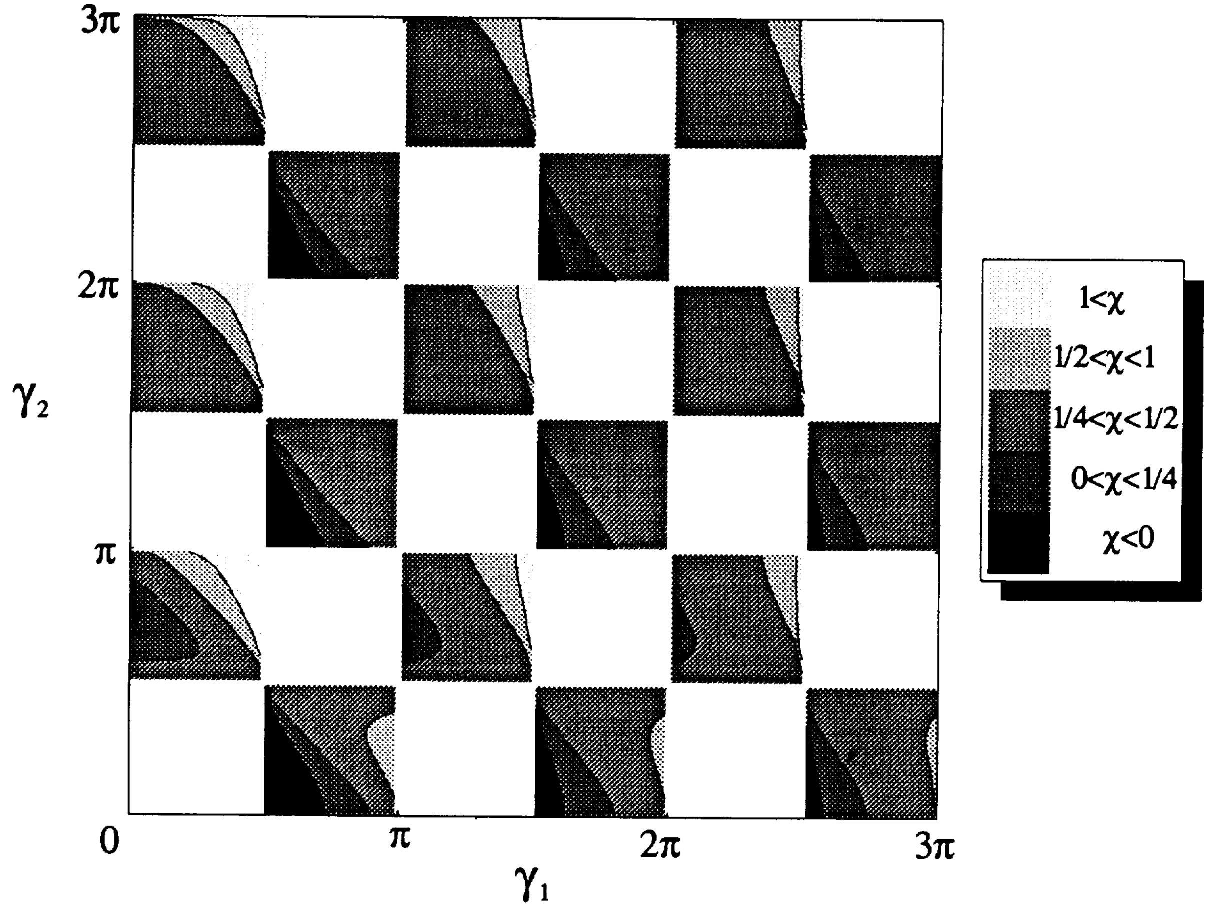

FIG. 1. The "chessboard of distorted time" for the free evolution of a charged particle manipulated by the magnetic field (11–12). The level curves for the distorsion coefficient χ are ploted as functions of the angles γ_1 and γ_2 and mark the zones for which at $\tau = 6T$ we obtain the acceleration ($\chi > 1$), slowing ($0 < \chi \leq 1$) or regression ($\chi \leq 0$) of the free evolution.

As a final remark we would like to point out that this kind of manipulations is not restricted to rectangular pulses of magnetic field. The same possibilities can be found if smooth fields are used, although we won't have analytic expressions for the discriminant anymore.

Acknowledgments

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SECTION 3

AMPLIFIER AND CAVITY ELECTRODYNAMICS